

# Cluster Distribution in Mean-Field Percolation: Scaling and Universality

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## Abstract

The partition function of the finite  $1 + \epsilon$  state Potts model is shown to yield a closed form for the distribution of clusters in the immediate vicinity of the percolation transition. Various important properties of the transition are manifest, including scaling behavior and the emergence of the spanning cluster.

## I. INTRODUCTION

One of the aspects of bond percolation that has captured the imagination of researchers is the collection of scaling properties that a percolating system exhibits in the vicinity of the transition at which the spanning cluster emerges [1]. These scaling properties manifest themselves in various correlation functions, which reveal the structure of large finite clusters, and through the moments of the cluster size distribution function. An adjunct of these scaling properties are quantities that are *universal* at the percolation transition. Universality reflects the insensitivity of behavior at and near the transition to details, and follows from the dominating influence of long range correlations. The key feature of the percolation transition

is the emergence of the spanning cluster, and the concepts of scaling and universality are of great help in the description of its characteristics in the vicinity of the transition.

Many of the scaling properties of the percolation transition, and all information regarding the moments of the cluster distribution, are contained in the generating function, defined in terms of the distribution of cluster sizes as follows

$$F(p, h) = \sum_m n_m^c e^{-mh} \quad (1.1)$$

In the above, the quantity  $n_m^c$  is the average number of clusters containing  $m$  sites, and  $p$  quantifies the probability that a link between sites is “active.” An active link is called a bond. In the simplest versions of bond percolation only those links that couple nearest neighbor sites will, with any finite probability, be bonds.

In the standard version of percolation, only bonds connecting close-by sites are allowed to be active. A version of percolation that lends itself to exact analysis is the infinite range model investigated by Wu [2] and, as an example of a random graph, by Erdős and Rényi [3]. In this model the probability that a bond exists between two sites is independent of the distance from one to the other. If there are  $N$  sites, the probability that a bond exists between any given pair is equal to  $p/N$ . In the “thermodynamic limit”  $N \rightarrow \infty$  this quantity vanishes, but the effective coordination number of each site diverges as  $N$ , and the net probability that two sites are connected does not necessarily approach zero, or any other trivial value.

The infinite range model exhibits a percolation transition, in that the probability,  $P$ , that two arbitrarily chosen sites belong to the same cluster, which is equal to zero in the thermodynamic limit when  $p \leq 1$ , takes on a finite value when  $p > 1$ . The equation satisfied by the quantity  $P$  is

$$P = 1 - e^{-pP}. \quad (1.2)$$

The only non-negative solution to this equation is  $P = 0$  when  $p \leq 1$ , while a non-zero solution exists when  $p > 1$ . This latter solution saturates at 1 in the limit  $p = \infty$ . The

quantity  $P$  is also equal to the fraction of sites contained in the spanning cluster. This cluster contains a finite fraction of all the sites in the system in the thermodynamic limit.

The connection between the generating function and the statistical mechanics of a particular model was established by Fortuin and Kasteleyn [4], who demonstrated the equivalence between the percolation generating function and the partition function of the  $q \rightarrow 1$  limit of the  $q$ -state Potts model [5]. In particular, the generating function as given by Eq. (1.1) is, to within uninteresting factors, equal to the limiting ratio

$$\lim_{q \rightarrow 1} \frac{Z_q - Z_1}{q - 1}, \quad (1.3)$$

where  $Z_q$  is the partition function of the  $q$ -state Potts model. A number of field-theoretical treatments of percolation are based on the above relation [6].

Given the generating function one can, in principle, determine the values of the quantities  $n_m^c$ . This is because the sum in Eq. (1.1) has the general form of a Laplace transform, and such transforms are readily inverted. Given the dependence on  $h$  of  $F(p, Hh)$  one obtains the mean number of clusters containing  $n$  sites via

$$n_m^c = \frac{1}{2\pi} \int_0^{2\pi} F(p, ih) e^{-ihm} dh \quad (1.4)$$

In this brief note we show that results previously obtained for the partition function of the  $1 + \epsilon$ -state Potts model with infinite range interactions lead directly to the cluster distribution function for the infinite-range version of percolation described above. This distribution function displays all the expected scaling properties, and in addition reveals the precise way in which the spanning cluster emerges from the “sea” of clusters that remain finite in the thermodynamic limit. As the model investigated is the “mean field” version of short-ranged percolation, the results to be displayed for the cluster size distribution represent zeroth order approximations to the corresponding results relating to the cluster size distribution function of the physically realizable, and therefore physically relevant, short-ranged percolation [7].

## II. SCALING AT THE PERCOLATION TRANSITION

The scaling laws that characterize the critical point have direct analogues in the percolation transition. For instance, given the critical value of  $p$ , which we denote  $p_c$ , the percolation generating function in the immediate vicinity of the transition takes on the following form

$$F(p_c(1 + \Delta p), h) \rightarrow |\Delta p|^{w_1} f\left(|\Delta p|^{-w_2} h, \frac{\Delta p}{|\Delta p|}\right). \quad (2.1)$$

The exponents  $w_1$  and  $w_2$  control the asymptotic behavior of various aspects of the cluster size distribution. For example the  $l^{\text{th}}$  moment of the cluster distribution function, equal to the expectation value of the  $l^{\text{th}}$  moment of the cluster size  $m$ , is given by

$$\langle m^l \rangle = \frac{\sum_m m^l n_m^c}{\sum_m n_m^c} = \frac{(-1)^l \frac{d^l}{dh^l} F(p, h) \Big|_{h=0}}{F(p, 0)}, \quad (2.2)$$

as can be established by looking at Eq. (1.1).

Given the scaling form in Eq. (2.1), the dependence of the expectation value of  $m^l$  on  $\Delta p$  is

$$\langle m^l \rangle \propto |\Delta p|^{-lw_2}. \quad (2.3)$$

The constant of proportionality depends on sign of  $\Delta p$ . Eliminating the denominator in Eqs. (2.2), we have the following relations

$$\sum_n n_m^c m^l = \frac{d^l}{dh^l} F(p, h) \Big|_{h=0}. \quad (2.4)$$

when the power  $l$  is equal to one, the right hand side is just the total number of sites in the system, and this quantity is clearly independent of the clustering induced by the existence of bonds. This sum rule, which is violated in the thermodynamic limit if when  $h \rightarrow 0^-$  plays the role of the symmetry that is violated when there is a symmetry-breaking transition.

Now, the scaling form in Eq. (2.1), along with the inversion formula (1.4) implies a cluster size distribution having the following scaling form

$$n_m^c = |\Delta p|^{w_1+w_2} \mathcal{X}\left(m |\Delta p|^{w_2}, \frac{\Delta p}{|\Delta p|}\right). \quad (2.5)$$

At the percolation transition, where  $\Delta p = 0$ , Equation (2.5) implies a distribution function having the following form:

$$n_m^c \propto m^{-(w_1+w_2)/w_2}. \quad (2.6)$$

In the mean field limit, the two exponents  $w_1$  and  $w_2$  take on the following values:

$$w_1 = 3 \quad (2.7)$$

$$w_2 = 2 \quad (2.8)$$

Then, according to Eq. (2.6),  $n_m^c \propto m^{-5/2}$ .

When there is a finite number of sites in the system, the cluster size distribution incorporates the number of sites,  $N$ , by taking on the more general scaling form

$$n_m^c = |\Delta p|^{w_1+w_2} \mathcal{X} \left( m |\Delta p|^{w_2}, \frac{|\Delta p|^{-w_3}}{N}, \frac{\Delta p}{|\Delta p|} \right). \quad (2.9)$$

In the mean field limit, the exponent  $w_3$  is equal to 3. Eq. (2.9) implies that the relation (2.6) applies at the percolation transition until  $m \propto N^{2/3}$ .

### III. CLUSTER SIZE DISTRIBUTION

The distribution of cluster sizes in mean field percolation follows directly from a result for the ratio in Eq. (1.3). This result was based on an analysis of the mean field version of the  $q$  state Potts model in the limit  $q \rightarrow 1$  [8]. In the calculation leading to a closed form expression for the generating function of the mean field Potts model limits were taken in the proper order, although the final result was obtained with the use of non-rigorous arguments. Making the following replacements

$$p = 1 + N^{-1/3}t, \quad (3.1)$$

$$h = HN^{-2/3} \quad (3.2)$$

then the generating function takes the following form

$$F(p, h) \rightarrow \int_{-\infty}^{\infty} d\Delta \left( \left\{ \int_0^{\infty} \exp \left[ \frac{-(L-t)^3}{6} - \frac{t^3}{6} - \Delta L \right] dL \right\} \right. \\ \left. \times \Im \ln \left\{ \int_c \exp \left[ (\Delta + H)x + \frac{x^3}{6} \right] \right\} \right) + K_c. \quad (3.3)$$

The contour integration in Eq. (3.3) is over a contour in the complex  $x$  plane that extends from  $-\infty$  on the real axis to  $\infty$  along a curve making an angle of  $60^\circ$  with respect to the positive real axis. For details see [8].

The inversion of this function according to Eq. (1.4) is straightforward to carry out. Shifting the integration variable by  $H$ , rotating by  $90^\circ$  in the complex plane, multiplying by  $e^{-imh}$  and integrating, one obtains immediately

$$n_m^c = N^{-2/3} \int_{-\infty}^{\infty} d\Delta \left( \exp \left[ \frac{-(mN^{-2/3} - t)^3}{6} - \frac{t^3}{6} - \Delta mN^{-2/3} \right] \right. \\ \left. \times \Im \ln \left\{ \int_c \exp \left[ \Delta x + \frac{x^3}{6} \right] \right\} \right). \quad (3.4)$$

Expression (3.4) embodies the full expected scaling form of the cluster size distribution, and represents the mean field limit of the distribution of cluster sizes in the case of short range bond percolation. As such, it ought to yield the distribution of cluster sizes on a lattice in more than six dimensions, six being the upper critical dimension for short range bond percolation [6]. In addition, it constitutes the “zeroth order” distribution, about which one perturbs to obtain the cluster size distribution in bond percolation in lower dimensionality.

As a test of the validity of Eq. (3.4), we have measured the distribution of cluster sizes for mean field bond percolation on systems with various numbers of sites,  $N$ . The results are displayed in Figs. 1 - 6. The fit between the simulations and (3.4) is excellent below the percolation transition, and when the cluster size is not too large. When  $t > 0$ , so that the threshold for percolation in the “thermodynamic limit” has been exceeded, a feature appears in the distribution in the form of a peak in the upper reaches of the distribution. This peak—which can be demonstrated to have an integrated weight of unity when  $t$  is large and positive—corresponds to the contribution to the distribution of what becomes the spanning cluster in the limit of an infinite system. As can be seen in Figure 5, perfect

agreement with simulations is not achieved for any of the systems explored. On the other hand, there is clear evidence for convergence between the expression (3.4) and the results of numerical calculations as the number of sites increases to fairly large values. We are confident that a system with the sufficient number of sites will have a cluster distribution that is governed by Eq. (3.4). At this point, we do not understand the reason for the slow approach of the data to what appears to be its proper limiting form.

The derivation of the form (3.3) of the generating function for percolation on a finite lattice was not entirely rigorous [8]. The test of the distribution in Eq. (3.4) can thus be regarded as a test of that form. Given the clear evidence for agreement between the predictions based on that form and the results of simulations, one there is increasing confidence for the validity of the arguments that underly it.

## REFERENCES

- [1] D. Stauffer and A. Aharony, **Introduction to Percolation Theory**, (Taylor & Francis, London, 1992), and references therein.
- [2] F. Y. Wu, *J. Phys.* **A15**, L365 (1982).
- [3] P. Erdős and A. Rényi, *Pub. Math. Inst. Hung. Acad. Sci.* **5**, 17 (1960).
- [4] C. M. Fortuin and P.W. Kasteleyn *Physica*, **57**, 536 (1971).
- [5] R. B. Potts, *Proc. Cambridge Philos. Soc.*, **48**, 106 (1952).
- [6] A. B. Harris, T. C. Lubensky, W. K. Holcomb and C. Dasgupta *Phys. Rev. Lett.* **35**, 327 (1975).
- [7] See, for example J. Rudnick, J. Morris and G. Gaspari, *Phys. Rev. B* **49**, 1016 (1994).
- [8] J. Rudnick and G. Gaspari *J. Stat. Phys.* **42**, 833 (1986).



## FIGURES

FIG. 1. The cluster size distribution,  $n_m^c$ , multiplied by  $N^{2/3}$ , plotted against  $mN^{-2/3}$ , where  $m$  is the size of the cluster and  $N$  is the number of sites in the system. The graph in this Figure is for  $t = -1$ , where the quantity  $t$  is defined in Eq. (3.1). The system is close to the percolation transition, but the transition has not yet been reached. Note the excellent agreement between the solid curve, representing the predictions of Eq. (3.4) and the results of simulations for  $N = 10,000$ , 40,000 and 400,000.

FIG. 2. The cluster size distribution when  $t = 0$ . In the bulk limit, this is the exact location of the percolation transition.

FIG. 3. The cluster size distribution when  $t = 1$ . The system is just above the percolation transition, and the incipient spanning cluster has begun to emerge. The signature in the distribution function is a barely visible feature.

FIG. 4. The cluster distribution when  $t = 2$ . Now, the peak for the spanning cluster is becoming distinct. Agreement between the analytical prediction and the results of simulations is not nearly as good in the vicinity of this peak as elsewhere in the Figure. However, agreement improves with increased system size

FIG. 5. The cluster distribution when  $t = 3$ . The spanning cluster peak is well separated from the rest of the distribution. Agreement between analysis and simulations is not good in the vicinity of the peak, but, as previously, it improves with increasing system size. The tendency strongly indicates convergence.

FIG. 6. a log-log plot of the distribution at the percolation transition ( $t = 0$ ). In the infinite system, this plot would have the form of a straight line. In the finite system, a power law is obeyed until  $m \propto N^{2/3}$ . This behavior is evident in the Figure, and is displayed by both the analytical form and the results of simulations. As in the previous Figures, the agreement between analysis and simulations is best for the largest systems.











